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INVOLUTORY TRANSFORMATIONS IN THE PROJECTIVE GROUP AND IN ITS SUBGROUPS*

BY EDWIN BIDWELL WILSON

THE importance of simple transpositions in the general theory of substitutions would naturally lead to the investigation of the corresponding geometric problem of the relation of involutory transformations† to the groups in which they occur; and in fact a considerable number of researches upon this subject have been published. Several mathematicians, obviously under the influence of Cremona, have taken up the involutory transformations of the Cremona type in all generality, and Kantor has even extended the field of operations to Cremona transformations of period higher than two.‡ On the other hand Study and Wiener, working in the early nineties, chose a more special field and brought the theory of the group of motions and transformations of symmetry in its relation to the allied involutory transformations to a high state of detailed development. §

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* Presented to the American Mathematical Society, September 4, 1906, as the introduction to a paper on "Oblique reflections and unimodular strains."

† An involutory transformation may be defined as one of period 2; and a transformation is said to be of period p when p is the smallest number such that a p-fold repetition of the transformation gives the identical transformation, that is, $T^p = 1$ or $T^{p+1} = T$.

‡ After the initial work of Cremona and Jonquières, there followed a large number of papers on periodic and in particular involutory bi-rational transformation. It will be sufficient to cite memoirs by Bertini in the *Rendiconti of the Royal Institute of Lombardy* for the years 1883 and 1889, where ample references to preceding work are given: and Kantor's *Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene*, Berlin, 1895.

§ E. Study, "Von den Bewegungen und Umlegungen," *Mathematische Annalen*, vol. 39, pp. 441-566 (1891). H. Wiener, a series of papers in the *Leipziger Berichte*, vols. 42, 43, 45 (1890-93).

This work was taken up later by Gale and Wood, the latter of whom extended it to cover non-Euclidean motions.* Smith then attacked the classical problem of the transformation of a quadratic form into itself, and solved it by means of central reflections.† If the quadratic surface obtained by setting the form equal to zero be considered as the absolute in a non-Euclidean geometry, it appears that the work done by Smith is, in its geometrical aspect, still essentially confined to the group of (non-Euclidean) motions and transformations of symmetry (in any number of dimensions).

In regard to the study of the general group of collineations from the point of view of its involutory transformations very little of a detailed nature has been accomplished. Some years ago I published a short note concerning the relation of the general collineation in space of three dimensions to skew reflections;‡ but as Smith subsequently showed, this question is much better treated by introducing line coordinates and regarding the problem as one in non-Euclidean geometry of five dimensions.§ After introducing the conception of area (or volume) relative to a line (or plane), I treated in some detail those collineations of the plane which leave invariant the area relative to some line.‖ Recently Haskell has obtained an existence theorem for the decomposition into central reflections of any n–dimensional collineation which has a complete set of fixed points; but he leaves the geometry of the problem quite untouched.¶

The object of the present communication is to set forth systematically certain elementary properties of involutory projective transformations which are at the basis of the investigations mentioned above and which shall serve as an introduction to further work. It may be stated that there is an algebraic as well as geometric interest which the present problem may serve. The transformation

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* A. S. Gale, "Wiener's theory of displacements, etc.," ANNALS OF MATHEMATICS, ser. 2, vol. 2, pp. 1–7 (1900). R. G. Wood, "The collineations of space which transform a nondegenerate quadric surface into itself," *ibid.*, pp. 161–171 (1901).

† P. F. Smith, "On the linear transformations of a quadric into itself," *Trans. Amer. Math. Soc.*, vol. 6, pp. 1–16 (1905).

‡ E. B. Wilson, *Trans. Amer. Math. Soc.*, vol. 1, pp. 193–196 (1900).

§ P. F. Smith, *loc. cit.*, p. 15.

‖ E. B. Wilson, "Ueber eine vom Begriffe der Länge unabhängige Definition des Volumens," *Jahresbericht der Deutschen Mathematiker-Vereiningung*, vol. 12, pp. 555–561 (1903); and "A generalised conception of area; applications etc.," ANNALS OF MATHEMATICS, ser. 2, vol. 5, pp. 29–45 (1903).

¶ M. W. Haskell, "The resolution of any collineation into perspective reflections," *Trans. Amer. Math. Soc.*, vol. 7, pp. 361–369 (1906).

$$\begin{array}{l} \rho x_1' = a_{11} x_1 \quad + \cdots + a_{1\,n+1} \quad x_{n+1} \\ \rho x_2' = a_{21} x_1 \quad + \cdots + a_{2\,n+1} \quad x_{n+1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \rho x_n' = a_{n1} x_1 \quad + \cdots + a_{n\,n+1} \quad x_{n+1} \\ \rho x_{n+1}' = a_{n+1\,1} x_1 + \cdots + a_{n+1\,n+1} x_{n+1} \end{array}$$

may be regarded in a double aspect. If the x's are interpreted as homogeneous point coordinates in space of n dimensions, the equations are those of the general projective transformation in this space; but if the x's are interpreted as Cartesian coordinates of the points in space of $n + 1$ dimensions (and a fixed value, say unity, is given to ρ) the equations become those of the general homogeneous strain in $n + 1$ dimensions. This strain may be regarded as a multiple quantity in $(n + 1)^2$ units. The product of two strains will be given by the properly chosen product of the corresponding multiple quantities.* As the identical transformation is represented by the idemfactor I, the involutory transformations must be the geometric counterparts of the square roots of I (or, in the projective group, of kI).

Thus, *the involutory transformations of the linear group are interesting not only because they are the projective generalisations of the various kinds of ordinary physical reflections and because they are the natural generalisations of simple transpositions, but also for the reason that they correspond to the simplest roots of the simplest multiple quantity—the square root of the idemfactor, which is but the generalisation of unity.*†

1. The projective involutory transformations. It is well known that the various physical reflections and their projective generalisations are involutory collineations. It seems to be taken for granted that these are the only collineations which are involutory—which indeed is true. The proof of this fact has been given by Bertini;‡ but as no subsequent writers refer to his paper, a somewhat different demonstration of the theorem may be given here for the sake of completeness and in order that we may have more definitely in mind the fundamental geometric entity with which we deal. The proof is by induction, and depends on the theorem that any collineation in n-dimensional space has at least one fixed linear space of $n - 1$ dimensions.* (If it were assumed that an involutory transformation had a complete set of fixed points, that is, in space of n dimensions $n + 1$ linearly independent fixed points, the proof would be immediate and obvious. It is, however, our purpose throughout this work to avoid the vague assumption that we are dealing with the "general" case—an assumption which is the bane of a great deal of geometric work).†

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* See Gibbs, "On multiple algebra," *Proceedings of the American Association for the Advancement of Science*, vol. 35, 1886.

† The detailed development of this aspect of the relation of involutory transformations to the general projective group is reserved for a later paper on "Oblique reflections and unimodular strains."

‡ Bertini, in the *Rendiconti of the Royal Institute of Lombardy*, vol. 19, pp. 176–183 (1886) under the title "Le omografie involutorie in uno spazio lineare a qualsivoglia numero di dimensioni."

Consider first the case in one dimension. By virtue of the fundamental theorem which has been assumed, any linear transformation of a line must have one fixed point. If this be taken as the point "infinity" of a projective number system upon the line, the transformation becomes

$$T: \qquad x' = ax + b$$

and

$$T^2: \qquad x'' = ax' + b = a^2x + b(a + 1) = x.$$

Hence

$$a^2 = 1 \quad \text{and} \quad b(a + 1) = 0.$$

If now $a = +1$, it follows that $b = 0$, and hence T reduces to $x' = x$ which is the identical transformation, obviously though trivially involutory. If on the other hand $a = -1$, a change of the origin to the point $b/2$ brings the equation of transformation to the form

$$T: \qquad x' = -x \qquad \text{or} \qquad \rho x_1' = -x_1, \qquad \rho x_2' = +x_2.\ddagger$$

Thus the transformation has two fixed points, the origin and infinity, and is merely the well known involution on a line. *A similar result obviously*

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* See Muth, *Theorie der Elementartheiler*, p. 200. Or better, merely note that if the x''s be set equal to the x's in the equations of the transformation given above and if the condition be expressed that the resulting $n + 1$ equations have a solution for the x's, a value of ρ may be obtained which will satisfy the equations and thus a fixed point of the transformation is found. By duality, there must be a fixed linear space of $n - 1$ dimensions.

† See "A problem in analytic geometry, with a moral," by M. Bôcher, ANNALS OF MATHEMATICS, ser. 2, vol. 7, pp. 44–49 (1905).

‡ A proof quite similar to this may be applied to any periodic transformation of order p. In fact the formulas for T^p are: $x^{(p)} = a^p x + b(a^{p-1} + a^{p-2} + \cdots + a + 1) = x$. Hence a is some p^{th} root of unity. If $b = 0$, T is already of the type desired; if not, a transformation of the origin to the point $b/(1 - e)$, where e is the p^{th} root of unity, brings T to the type $\rho x_1' = ex_1$, $\rho x_2' = x_2$. Hence there are two fixed points.

holds in the case of a pencil of lines or of planes, or any set which depends on one linear parameter and which undergoes an involutory transformation.

In the case of the plane one line must be fixed. It may be fixed point for point or only as a whole—in which case the transformation upon it must be involutory. In either case there are at least two distinct fixed points on the line. Consider the pencils which have their vertices at these points. Each of them is transformed either identically or in an involutory fashion. Hence, as has been seen above, there must be not less than two fixed rays in each pencil, one in addition to the assumed fixed line. These additional rays intersect in a point which is fixed. Hence in the plane the involutory transformation has a complete set of fixed points. Passing on to space we note that there must be at least one fixed plane by virtue of the assumed fundamental theorem. Whether this plane is transformed identically or in involutory fashion, it contains a complete system of three fixed points joined by three fixed lines. If the same reasoning be applied to the pencils of planes which have these lines as axes as was applied to the pencils in the case of the plane, it becomes evident that there is a fourth fixed point not in the plane which was assumed as fixed. Hence the complete set of points for space; and so on to any number of dimensions.

The equations of any collineation in n dimensions referred to a system of fixed points are

$$\begin{array}{rlcrl} T: & \rho x_1' = a_1 x_1 & \qquad & T^2: & \rho x_1'' = a_1^2 x_1 \\ & \rho x_2' = a_2 x_2 & & & \rho x_2'' = a_2^2 x_2 \\ & \cdots\cdots & & & \cdots\cdots \\ & \cdots\cdots & & & \cdots\cdots \\ & \rho x_n' = a_n x_n & & & \rho x_n'' = a_n^2 x_n \\ & \rho x_{n+1}' = a_{n+1} x_{n+1} & & & \rho x_{n+1}'' = a_{n+1}^2 x_{n+1}. \end{array}$$

Hence if T is involutory the a's are square roots of unity and the equations may be written in the form

$$\rho x_1' = \pm x_1,\ \rho x_2' = \pm x_2,\ \cdots\ \rho x_n' = \pm x_n,\ \rho x_{n+1}' = \pm x_{n+1}.^*$$

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* The proof may be extended to transformations of period p. No step has to be altered, except that the preceding foot-note must replace the work in the text. The result is that any transformation of period p may be written in the form

$$\rho x_1' = e_1 x_1,\ \rho x_2' = e_2 x_2,\ \ldots,\ \rho x_n' = e_n x_n,\ \rho x_{n+1}' = e_{n+1} x_{n+1},$$

where $e_1, e_2, \ldots, e_{n+1}$ are any p^{th} roots of unity.

There are many different combinations of sign which are possible, but owing to the possibility of permuting the letters and of multiplying through by a factor, the number of essentially different involutory projective transformations is only $E\left(\frac{n+1}{2}\right)$, where $E(x)$ denotes the integral part of x. If the first k signs be taken negative and the remaining $n-k+1$ be positive, the transformation takes the form

$$\rho x_1' = -x_1,\ \rho x_2' = -x_2,\ \cdots,\ \rho x_k' = -x_k,$$

$$\rho x_{k+1}' = x_{k+1},\ \cdots,\ \rho x_n' = x_n,\ \rho x_{n+1}' = x_{n+1}.$$

The linear spaces defined by

$$S_{n-k}:\quad x_1 = x_2 = \cdots = x_k = 0,$$

$$S_{k-1}:\quad x_{k+1} = \cdots = x_n = x_{n+1} = 0$$

remain identically fixed, as the ratios of the coordinates of their points are not changed. The coordinates of any pair of points which interchange their positions are

$$x_1 : x_2 \quad : \quad \cdots \quad : \quad x_k : x_{k+1} : \quad \cdots \quad : x_n : x_{n+1},$$

$$-x_1 : -x_2 : \quad \cdots \quad : -x_k : x_{k+1} : \quad \cdots : \quad x_k : x_{k+1}.$$

These points are collinear with and harmonically situated with repect to the two points

$$0 : 0 : \quad \cdots \quad : 0 : x_{k+1} : \quad \cdots \quad : x_n : x_{n+1},$$

$$x_1 : x_2 : \quad \cdots \quad : x_k : \ 0 \ : \quad \cdots \quad : 0 : \ 0,$$

which lie respectively in the spaces S_{n-k} and S_{k-1}. As this is precisely the definition of a projective reflection, *it appears that the only involutory collineations are the projective reflections of the* $E\left(\frac{n+1}{2}\right)$ *different types above found.* That the projective reflection of the points of S_n in any two non-intersecting spaces S_{k-1} and S_{n-k} is involutory and definite, is obvious.

If the symbol $T(k-1, n-k)$ denote a projective reflection of S_n in the two non-intersecting spaces S_{k-1} and S_{n-k}, the number of degrees of freedom of T is the sum of the numbers of degrees of freedom of the spaces

S_{k-1} and S_{n-k} in S_n. Now the number of degrees of freedom of S_i in S_n is

$$n(i+1) - (i+1)i = (n-i)(i+1),$$

because it takes $i+1$ points each with n coordinates to determine S_i, while at the same time each of these $i+1$ points has i degrees of freedom. Hence $T(k-1,\, n-k)$ has

$$(n-k+1)k + k(n-k+1) = 2k(n-k+1) = 2(k-1+1)(n-k+1)$$

degrees of freedom. The following table may therefore be constructed in which the first column gives the number of dimensions, the second the type of the reflection, the third its number of degrees of freedom, the fourth the number of degrees of freedom in the general collineation, the fifth the number of involutory collineations required for the resolution of the collineation, and the sixth the remaining degrees of freedom.

$n=1$	$T(0,0)$	2	3	2	1
$n=2$	$T(0,1)$	4	8	3	4
$n=3$	$T(0,2)$	6	15	4	9
	$T(1,1)$	8	15	3	9
$n=n$	$T(0,\, n-1)$	$2n$	$n(n+2)$	$n+1$	n^2
	$T(k-1,\, n-k)$	$2k(n-k+1)$	$n(n+2)$	?	?
n odd	$T\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$	$\frac{(n+1)^2}{2}$	$n(n+2)$	3 ?[*]	$\frac{(n+1)^2}{2}+1$
n even	$T\left(\frac{n}{2}-1, \frac{n}{2}\right)$	$\frac{n(n+2)}{2}$	$n(n+2)$	3 ?	$\frac{n(n+2)}{2}$

So much for the general theory.

2. The discussion of the subgroups. The number of types of involutory collineations in the general projective group has been seen to be

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[*] It seems probable that if the projective reflection which has the greatest number of degrees of freedom be chosen, three will suffice for the resolution of the general collineation. The proof has thus far been given only in the case of $n \leqq 3$; see p. 78.

$E\left(\frac{n+1}{2}\right)$. In the various subgroups the number of types may be the same or less or more. For each of these subgroups is specified by the addition of certain invariants. The problem of determining all the involutory transformations in any subgroup of the projective group is merely that of combining in all possible distinct ways the fixed spaces of the reflections with the invariants of the special group. This may be illustrated as follows.

In the group of motions in the plane the two circular points are individually fixed. Hence two cases may arise. Either the line at infinity is the fixed line of the reflection $T(0, 1)$ and the fixed point any point of the plane, or one of the circular points is the fixed point, that is, the center of the reflection, and the fixed line or axis of the reflection passes through the other. The first case gives rise to the ordinary physical reflection in a point or, in other words, to rotation about the point through the angle of 180°. The second case gives a transformation which may be written with the Cartesian equations

$$x' = iy, \quad y' = -ix, \quad i = \sqrt{-1}.$$

The use of this transformation for studying the Euclidean group seems slight. The product of any number of physical reflections in points produces merely a translation and therefore cannot suffice for the general discussion of the Euclidean group. It may be noted that the identical invariance of the circular points does not restrict the group to that of Euclidean motions, but allows dilatations to enter. As the reflections above mentioned cannot introduce any dilatation, it is not necessary to trouble with that additional invariant of the Euclidean group which serves to shut out the dilatations.

If the set of transformations of symmetry be adjoined to the Euclidean group, the resulting group leaves the circular points invariant merely as a pair. Hence in addition to the foregoing cases the center of the reflection may be any point on the line at infinity and the fixed line (which must cut the line at infinity in a point harmonically situated to the center with respect to the circular points) will take a direction perpendicular to the direction leading to that point which was chosen as center. The resulting involutory transformation is then merely common orthogonal or physical reflection in the fixed line. The relation of this reflection to the group in question has been taken up elsewhere.* Passing on to the more general group which leaves a specified line invariant (the line may be taken as the line at infinity and the group then becomes the affine group) we note that there are also in this case two kinds of reflection, central and oblique reflection. The former arises when the fixed line of the group coincides with the axis of the reflection; the second when the center is situated on the fixed line. As a matter of fact the group which may be made up of such reflections leaves area with respect to the fixed line invariant.*

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* See Study and Wiener, *loc. cit.*

In a similar manner in space of three dimensions there is seen to be only one type of reflection in the Euclidean group. This is orthogonal reflection in a line or rotation about the line through 180°. Thus the Euclidean group contains fewer types of involutory transformations than the general group of all collineations. On the other hand if the group be extended to include all transformations of symmetry, there are three types of reflections—reflection in a point and orthogonal reflections whether in a line or in a plane. In this case the total number is greater than in the projective group. These groups have all been treated by other authors.† The natural generalisation of the Euclidean group to the non-Euclidean deserves a word and may be mentioned in the general case of any number of dimensions. The necessary and sufficient condition that a projective reflection shall leave a quadric surface invariant is that the fixed spaces S_{k-1} and S_{n-k} be polar with respect to the surface. Thus the number of types of non-Euclidean involutory transformations is always the same as the number in the general projective group in the same number of dimensions.‡

The next step in the development of the theory of involutory projective transformations is concerned with the unimodular strains. The restriction to unimodular strains is a necessary consequence of the invariance of volume under involutory strain. As the plane at infinity and the origin are to remain fixed, three and only three distinct types of involutory strain exist. If the projective transformation be of type $T(0, 2)$ the fixed plane may be taken at infinity and the fixed point at the origin. This gives rise to reflection through the origin. The direction of all vectors issuing from the origin is reversed. It is also possible that the fixed plane pass through the origin while the fixed point is any point at infinity. In this case the transformation becomes oblique reflection across the plane and in the direction leading to the fixed center at infinity.

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* See Wilson, ANNALS OF MATHEMATICS, *loc. cit.*

† Study, Wiener, and Gale, *loc. cit.*

‡ See Smith and Wood, *loc. cit.*

If vectors be resolved into two components, one along the fixed plane of the reflection and the other along the direction leading to the center at infinity, the former components will remain unchanged while the latter are reversed in direction. If the projective reflection be of type $T(1, 1)$ one of the fixed lines may be arbitrarily chosen in the plane at infinity and the other must then pass through the origin with no other restriction save that it shall not be complanar with the first. If a plane be passed through the origin and the fixed line at infinity and if vectors be resolved parallel to this plane and to the fixed line passing through the origin, it is seen that the transformation consists in leaving the latter components unchanged and in reversing the directions of the former.*

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* The detailed treatment of the relation between these three involutory strains and the general unimodular strain is interesting from a number of points of view, geometric, algebraic, and physical. It will form the subject of another paper.